

FROM DIXMIER ALGEBRAS TO STAR PRODUCTS

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ABSTRACT. Let \mathcal{M} be a Galois cover of a nilpotent coadjoint orbit of a complex semisimple Lie group. We define the notion of a *perfect* Dixmier algebra for \mathcal{M} and show how this produces a graded (non-local) equivariant star product on \mathcal{M} with several very nice properties. This is part of a larger program we have been developing for working out the orbit method for nilpotent orbits.

1. INTRODUCTION

This note is a companion piece to [B1], and a bridge between the approaches to quantization of nilpotent orbit covers in [B1] and its sequel [B2]. See the introduction to [B1] for background, references and motivations.

The purpose of this note is to set out some basic relations between the representation theoretic notion of Dixmier algebras and the Poisson geometric notion of star products.

2. PERFECT DIXMIER ALGEBRAS

Let \mathcal{O} be a complex nilpotent orbit in some complex semisimple Lie algebra \mathfrak{g} . We assume \mathcal{O} spans \mathfrak{g} . Let G^{sc} be a simply-connected complex Lie group with Lie algebra \mathfrak{g} .

Let $\kappa : \mathcal{M} \rightarrow \mathcal{O}$ be a covering where \mathcal{M} is connected. Then the following geometric structure lifts from \mathcal{O} to \mathcal{M} : the adjoint action of G^{sc} , the KKS symplectic form, the Hamiltonian functions ϕ^x , $x \in \mathfrak{g}$ defined by $\phi^x(y) = (x, y)_{\mathfrak{g}}$, and the square of the Euler dilation action of \mathbb{C}^* . In fact, G acts on \mathcal{M} where G is the quotient of G^{sc} by the (finite central) subgroup which fixes \mathcal{M} .

Consider the algebra

$$\mathcal{R} = R(\mathcal{M}) \tag{2.1}$$

of regular functions on \mathcal{M} . Then the Euler grading $\mathcal{R} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \mathcal{R}^j$ makes \mathcal{R} into a graded Poisson algebra in the sense of [B1, Definition 2.2.1] with $\phi^x \in \mathcal{R}^1$, for $x \in \mathfrak{g}$, and $\{\phi^x, \phi^y\} = \phi^{[x, y]}$. Also \mathcal{R} is a superalgebra with even and odd parts $\mathcal{R}^{even} = \bigoplus_{j \in \mathbb{N}} \mathcal{R}^j$ and $\mathcal{R}^{odd} = \bigoplus_{j \in \mathbb{N} + \frac{1}{2}} \mathcal{R}^j$. Let α be the algebra automorphism of \mathcal{R} such that $\alpha(f) = i^{2j} f$ if $f \in \mathcal{R}^j$. Notice $\alpha^4 = 1$.

Suppose $m \in \mathcal{M}$ covers $e \in \mathcal{O}$. Then \mathcal{M} is a Galois cover of \mathcal{O} if and only if G^m is normal in G^e . In this event, $\mathcal{S} = G^e/G^m$ is the Galois group, \mathcal{S} acts on \mathcal{M} by symplectic automorphisms, and our grading of \mathcal{R} is \mathcal{S} -invariant. The universal cover of \mathcal{O} is always Galois.

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Now we strengthen the usual definition of Dixmier algebra (Vogan, McGovern) by adding three new axioms in (V), (VI) and (VII). We allowed redundancies in our axioms in order to make them as explicit as possible.

Definition 2.1. Assume the cover \mathcal{M} of \mathcal{O} is Galois with Galois group \mathcal{S} . A *perfect Dixmier algebra* for \mathcal{M} is a noncommutative algebra \mathcal{D} together with the following data:

- (I) An increasing algebra filtration $\mathcal{D} = \cup_{j \in \frac{1}{2}\mathbb{N}} \mathcal{D}_j$ such that $[\mathcal{D}_j, \mathcal{D}_k] \subseteq \mathcal{D}_{j+k-1}$ for all $j, k \in \frac{1}{2}\mathbb{N}$.
- (II) A representation of \mathcal{S} on \mathcal{D} by filtered algebra automorphisms.
- (III) A Lie algebra embedding $\psi : \mathfrak{g} \rightarrow \mathcal{D}_1^{\mathcal{S}}$ such that the representation of \mathfrak{g} on \mathcal{D} given by the derivations $a \mapsto \psi^x a - a\psi^x$ is locally finite and exponentiates to a representation of G on \mathcal{D} by algebra automorphisms.
- (IV) An \mathcal{S} -equivariant graded Poisson algebra isomorphism $\gamma : \text{gr } \mathcal{D} \rightarrow \mathcal{R}$ such that $\gamma(\mathbf{p}_1(\psi^x)) = \phi^x$ where $\mathbf{p}_1 : \mathcal{D}_1 \rightarrow \text{gr}_1 \mathcal{D}$ is the natural projection.
- (V) An \mathcal{S} -invariant filtered algebra anti-automorphism β of \mathcal{D} such that (a) $\beta(\psi^x) = \psi^{-x}$, (b) β induces $\gamma^{-1}\alpha\gamma$ on $\text{gr } \mathcal{D}$ and (c) $\beta^4 = 1$.

We impose two further axioms. To state these, we notice two useful consequences of (I)-(V). First there is a unique G -linear map $T : \mathcal{D} \rightarrow \mathbb{C}$ such that $T(1) = 1$. Second, \mathcal{D} has become a superalgebra with $(G \times \mathcal{S})$ -invariant filtered algebra \mathbb{Z}_2 -grading $\mathcal{D} = \mathcal{D}^{\text{even}} \oplus \mathcal{D}^{\text{odd}}$ where the summands are the ± 1 -eigenspaces of β^2 . An element $a \in \mathcal{D}$ is *superhomogeneous* if $a \in \mathcal{D}^{\text{even}}$, in which case $|a| = 0$, or $a \in \mathcal{D}^{\text{odd}}$, in which case $|a| = 1$. Now we require that

- (VI) T is a supertrace.
- (VII) The G -invariant supersymmetric bilinear pairing $\mathcal{P}(a, b) = T(ab)$ is non-degenerate on \mathcal{D}_j for each $j \in \frac{1}{2}\mathbb{N}$.

In (VI), T is a supertrace means that if a and b are superhomogeneous then

$$T(ab) = (-1)^{|a||b|} T(ba) \quad (2.2)$$

when a and b have same parity, while $T(ab) = 0$ when a and b have different parity. Axioms (IV) and (V) guarantee that T vanishes on \mathcal{D}^{odd} and so axiom (VI) amounts to (2.2).

To make sense of (VII), we notice if $L \subseteq \mathcal{D}$ is any β^2 -stable subspace then we have a notion of the \mathcal{P} -orthogonal subspace L^\perp (since the right and left orthogonal subspaces coincide). We then say \mathcal{P} is *non-degenerate* on L if $L^\perp \cap L = 0$. The pairing \mathcal{P} is *supersymmetric* in the sense that $\mathcal{D}^{\text{even}}$ and \mathcal{D}^{odd} are \mathcal{P} -orthogonal, \mathcal{P} is symmetric on $\mathcal{D}^{\text{even}}$ and \mathcal{P} is anti-symmetric on \mathcal{D}^{odd} . Notice that (VII) is much stronger than saying \mathcal{P} is non-degenerate on \mathcal{D} .

We often speak of \mathcal{D} as the perfect Dixmier algebra, with the additional data being understood. See [B1, §7-8] for examples.

Here are some consequences of the axioms. First \mathcal{R} is \mathbb{N} -graded if and only if $\mathcal{D} = \mathcal{D}^{\text{even}}$. Indeed, \mathcal{R} is \mathbb{N} -graded $\Leftrightarrow \alpha^2 = 1 \Leftrightarrow \beta^2 = 1$, where the last equivalence follows by axiom (V)(b). An instance where \mathcal{R} is \mathbb{N} -graded is when $\mathcal{M} = \mathcal{O}$. Second, $\mathcal{D}^{\mathcal{S}}$ is a perfect Dixmier algebra for \mathcal{O} . This follows as $\mathcal{R}^{\mathcal{S}} = R(\mathcal{O})$ and all the Dixmier algebra data for \mathcal{M} is \mathcal{S} -equivariant.

Third, axiom (IV) provides a “symbol calculus” for \mathcal{D} with values in \mathcal{R} . If $a \in \mathcal{D}_j$ then the γ -symbol of a of order j is the image of a under the map

$$\mathcal{D}_j \rightarrow \mathcal{D}_j / \mathcal{D}_{j-\frac{1}{2}} \rightarrow \mathcal{R}^j$$

Fourth, let us extend $\psi : \mathfrak{g} \rightarrow \mathcal{D}_1^{\mathcal{S}}$ to an algebra homomorphism

$$\psi : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}^{\mathcal{S}} \quad (2.3)$$

and let J be the kernel of (2.3). Then $J \cap \mathcal{Z}(\mathfrak{g})$ is a maximal ideal of $\mathcal{Z}(\mathfrak{g})$ where $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathcal{U}(\mathfrak{g})$. This follows since by axioms (III) and (IV) the vector spaces \mathfrak{D} , $\text{gr } \mathfrak{D}$ and $R(\tilde{\mathcal{O}})$ are all \mathfrak{g} -isomorphic and so in particular $\mathfrak{D}^G = \mathbb{C}$.

Now (V)(a) says the following square is commutative:

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) & \xrightarrow{\psi} & \mathcal{D} \\ \downarrow \tau & & \downarrow \beta \\ \mathcal{U}(\mathfrak{g}) & \xrightarrow{\psi} & \mathcal{D} \end{array} \quad (2.4)$$

where τ is the principal anti-automorphism of $\mathcal{U}(\mathfrak{g})$. Consequently J is a τ -stable 2-sided ideal in $\mathcal{U}(\mathfrak{g})$. Furthermore J is a completely prime primitive ideal in $\mathcal{U}(\mathfrak{g})$. ($\mathcal{U}(\mathfrak{g})/J$ is a subalgebra of \mathcal{D} and so has no zero-divisors. This means J is completely prime. But also $J \neq \mathcal{U}(\mathfrak{g})$ and J contains a maximal ideal of the center of $\mathcal{U}(\mathfrak{g})$ and so, by a result of Dixmier, J is primitive.)

3. SIMPLICITY OF \mathcal{D}

Lemma 3.1. *Suppose axioms (I)-(VI) are satisfied. Let \mathcal{C} be some β^2 -stable subalgebra of \mathcal{D} . Then the following two conditions are equivalent:*

- (i) \mathcal{C} is a simple ring
- (ii) \mathcal{P} is non-degenerate on \mathcal{C}

Proof. (i) \Rightarrow (ii): \mathcal{C} is simple implies $\mathcal{C}^\perp \cap \mathcal{C} = 0$ since $\mathcal{C}^\perp \cap \mathcal{C}$ is a 2-sided ideal in \mathcal{C} which does not contain 1.

(ii) \Rightarrow (i): Let \mathcal{I} be a non-zero two-sided ideal in \mathcal{D}' . Pick $a \in \mathcal{I}$ with $a \neq 0$. Then (ii) implies there exists $b \in \mathcal{C}$ such that $T(ab) = 1$. It follows that $ab = c + 1$ where c lies in $\text{Ker } T \cap \mathcal{C}$. So \mathcal{I} contains $c + 1$. But then \mathcal{I} contains the G -subrepresentation generated by $c + 1$. Since $\text{Ker } T$ contains no non-zero G -invariants it follows (by complete reducibility of \mathcal{C} as a G -representation), that \mathcal{I} contains both c and 1. Thus \mathcal{C} is simple. \square

Proposition 3.2. *Suppose we are in the situation of Definition 2.1 and axioms (I)-(VI) are satisfied. Then \mathcal{D} is a simple ring if and only if $\mathcal{D}^{\mathcal{S}}$ is a simple ring.*

Proof. Suppose \mathcal{D} is simple. Then \mathcal{P} is non-degenerate on \mathcal{D} (by Lemma 3.1) and hence (since \mathcal{P} is \mathcal{S} -invariant) \mathcal{P} is non-degenerate on $\mathcal{D}^{\mathcal{S}}$. Then (by Lemma 3.1 again) $\mathcal{D}^{\mathcal{S}}$ is simple.

Conversely, assume $\mathcal{D}^{\mathcal{S}}$ is simple. Let \mathcal{I} be a non-zero 2-sided ideal in \mathcal{D} . To show \mathcal{D} is simple, it suffices to show that $\mathcal{I}^{\mathcal{S}} = \mathcal{I} \cap \mathcal{D}^{\mathcal{S}}$ is non-zero. Let $a \in \mathcal{I}$. Consider $b = (n!)^{-1} \sum_{\sigma \in S_n} (s_{\sigma_1} a) \cdots (s_{\sigma_n} a)$ where s_1, \dots, s_n is some listing of the elements of the

Galois group \mathcal{S} . Clearly $b \in \mathcal{I}^{\mathcal{S}}$. But also we can see using axiom (IV) that $b \neq 0$. Indeed, let j be the filtration order of a in \mathcal{D} and let $\phi \in \mathcal{R}^j$ be the γ -symbol of order j of a . Then b lies in \mathcal{D}_{jn} and the γ -symbol of order jn of b is $(s_{\sigma_1}\phi) \cdots (s_{\sigma_n}\phi)$. This product is non-zero and so b must be non-zero. \square

Corollary 3.3. *If \mathcal{D} is a perfect Dixmier algebra then \mathcal{D} and $\mathcal{D}^{\mathcal{S}}$ are both simple rings.*

Corollary 3.4. *If \mathcal{D} is a perfect Dixmier algebra and (2.3) is surjective, then the kernel J is a maximal ideal in $\mathcal{U}(\mathfrak{g})$.*

Remark 3.5. Ideally the axioms for a perfect Dixmier algebra should automatically imply that the image of (2.3) is simple, i.e., that J is maximal. Our axioms (I)-(VII) may not do this, but in any event our enriched axiom set (see Remark 4.4) accomplishes this.

4. THE NONCOMMUTATIVE \circ PRODUCT

By axiom (VII) in Definition 2.1, we have a unique \mathcal{P} -orthogonal decomposition

$$\mathcal{D} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \mathcal{D}^j \quad (4.1)$$

such that $\mathcal{D}_k = \bigoplus_{j=0}^k \mathcal{D}^j$. Each space \mathcal{D}^j is $(G \times \mathcal{S})$ -stable. It is easy to see that

$$\mathcal{P}(a, b) = \mathcal{P}(\beta(b), \beta(a))$$

for all $a, b \in \mathcal{D}$. Consequently \mathcal{D}^j is β -stable and using axiom (V)(b) we find that β acts on \mathcal{D}^j by multiplication by i^{2j} . Then $\mathcal{D}^{even} = \bigoplus_{j \in \mathbb{N}} \mathcal{D}^j$, $\mathcal{D}^{odd} = \bigoplus_{j \in \mathbb{N} + \frac{1}{2}} \mathcal{D}^j$ and T is the orthogonal projection of \mathcal{D} onto $\mathcal{D}^0 = \mathbb{C}$. Clearly there is a unique linear map

$$\mathbf{q} : \mathcal{R} \rightarrow \mathcal{D} \quad (4.2)$$

such that \mathbf{q} lifts $\gamma^{-1} : \mathcal{R} \rightarrow \text{gr } \mathcal{D}$ and $\mathbf{q}(\mathcal{R}^j) = \mathcal{D}^j$ for all $j \in \frac{1}{2}\mathbb{N}$. Then \mathbf{q} is a $(G \times \mathcal{S})$ -linear vector space isomorphism, and we have $\psi^x = \mathbf{q}^{-1}\phi^x$ and $\beta = \mathbf{q}\alpha\mathbf{q}^{-1}$. Now we define, for all $\phi, \psi \in \mathcal{R}$,

$$\phi \circ \psi = \mathbf{q}^{-1}((\mathbf{q}\phi)(\mathbf{q}\psi)) \quad (4.3)$$

Proposition 4.1. *Assume we have a perfect Dixmier algebra \mathcal{D} for \mathcal{M} with \mathcal{D}^j and \mathbf{q} defined as above. Then \circ is a $(G \times \mathcal{S})$ -invariant associative product on \mathcal{R} and so $(\mathcal{R}, +, \circ)$ is an associative noncommutative algebra. Then*

$$\mathcal{R}^j \circ \mathcal{R}^k \subseteq \mathcal{R}^{j+k} \oplus \mathcal{R}^{j+k-1} \oplus \cdots \oplus \mathcal{R}^{|j-k|} \quad (4.4)$$

where $j, k \in \frac{1}{2}\mathbb{N}$. Suppose $\phi \in \mathcal{R}^j$ and $\psi \in \mathcal{R}^k$ so that $\phi \circ \psi = \sum_p C_p(\phi, \psi)$ where $C_p(\phi, \psi)$ lies in \mathcal{R}^{j+k-p} . Then

$$\phi \circ \psi \equiv \phi\psi + \frac{1}{2}\{\phi, \psi\} \pmod{\mathcal{R}^{\leq j+k-2}} \quad (4.5)$$

$$C_p(\phi, \psi) = (-1)^p C_p(\psi, \phi) \quad (4.6)$$

Proof. The second sentence is clear. Now (4.4) is equivalent to

$$\mathcal{D}^j \mathcal{D}^k \subseteq \mathcal{D}^{j+k} \oplus \mathcal{D}^{j+k-1} \oplus \dots \oplus \mathcal{D}^{|j-k|} \quad (4.7)$$

We have $\mathcal{D}^j \mathcal{D}^k \subseteq \bigoplus_{i \in \mathbb{N}} \mathcal{D}^{j+k-i}$ since \mathcal{D} is a superalgebra. Because of axiom (VII), showing (4.7) reduces to showing that $\mathcal{D}^j \mathcal{D}^k$ is orthogonal to \mathcal{D}^s if $s < |j-k|$. So suppose $\mathcal{D}^j \mathcal{D}^k$ is not orthogonal to \mathcal{D}^s . Then there exist $a \in \mathcal{D}^j$, $b \in \mathcal{D}^k$ and $c \in \mathcal{D}^s$ such that $T(abc) = 1$. Then bc has a component in \mathcal{D}^j and so $k+s \geq j$. But also $T(bca) = \pm 1$ (since T is a supertrace) and so similarly $s+j \geq k$. Hence $s \geq |j-k|$. Now for $\phi \in \mathcal{R}^j$ and $\psi \in \mathcal{R}^k$ we can write

$$\phi \circ \psi = \sum_{p=0}^{2 \min(j,k)} C_p(\phi, \psi) \quad (4.8)$$

where $C_p(\phi, \psi) \in \mathcal{R}^{j+k-p}$.

Now axiom (IV) implies that $C_0(\phi, \psi) = \phi\psi$ and $C_1(\phi, \psi) - C_1(\psi, \phi) = \{\phi, \psi\}$. But also axiom (V) implies that the map $\alpha : \mathcal{R} \rightarrow \mathcal{R}$ (which is an algebra automorphism with respect to the ordinary product) is an algebra anti-automorphism with respect to \circ . Thus

$$\alpha(\phi \circ \psi) = (\alpha\psi) \circ (\alpha\phi) = i^{2j+2k} \psi \circ \phi$$

Then $i^{-2p} C_p(\phi, \psi) = C_p(\psi, \phi)$. This proves (4.6). Then in particular $C_1(\phi, \psi) = -C_1(\psi, \phi)$. So $C_1(\phi, \psi) = \frac{1}{2}\{\phi, \psi\}$ and we get (4.5). \square

We can think of $\mathbf{q} : \mathcal{R} \rightarrow \mathcal{D}$ a “quantization map” as in [B1, §8.1-8.2]. In particular we have

Corollary 4.2. *If $\phi \in \mathcal{R}^1$, for instance if $\phi = \phi^x$ where $x \in \mathfrak{g}$, then for all $\psi \in \mathcal{R}$ we have $\{\phi, \psi\} = \phi \circ \psi - \psi \circ \phi$.*

Proof. Identical to the proof of [B1, Corollary 8.2.3]. \square

Proposition 4.1 implies in particular that \mathcal{R} , equipped with its \circ product, *becomes the perfect Dixmier algebra!* The data on \mathcal{D} required by the axioms corresponds under \mathbf{q} to data that exist from the beginning on \mathcal{R} :

- (I) the grading on \mathcal{R} gives rise to the filtration by subspaces $\mathcal{R}^{\leq j}$,
- (II) the ψ^x correspond to the momentum functions ϕ^x of the Hamiltonian \mathfrak{g} -symmetry,
- (III) the Galois group \mathcal{S} already acts on \mathcal{R} ,
- (IV) γ corresponds to the identity map,
- (V) β corresponds to α .

Notice that the even and odd parts of \mathcal{D} correspond to the even and odd parts of \mathcal{R} . Also T corresponds to the projection

$$\mathcal{T} : \mathcal{R} \rightarrow \mathcal{R}^0 = \mathbb{C} \quad (4.9)$$

defined by the Euler grading. So $\mathcal{T}(\phi)$ is just the *constant term* of ϕ . The nondegenerate bilinear pairing \mathcal{Q} on \mathcal{R} corresponding to \mathcal{P} is given by $\mathcal{Q}(\phi, \psi) = \mathcal{T}(\phi \circ \psi)$. This pairing is *graded* supersymmetric in the sense that \mathcal{Q} pairs \mathcal{R}^j with \mathcal{R}^k trivially if $j \neq k$.

In this approach, the “new” axioms (VI) and (VII) have played a crucial role. Also we have gained a lot more structure on the Dixmier algebra. In particular the \circ product “breaks off” after degree $|j - k|$ in (4.4); this is a Clebsh-Gordan type phenomenon.

Thus the problem of finding a Dixmier algebra for \mathcal{M} can be reformulated as the problem of constructing a suitable product \circ on \mathcal{M} . To formalize this we make

Definition 4.3. Assume \mathcal{M} is a Galois cover of \mathcal{O} with Galois group \mathcal{S} . A *perfect Dixmier product* on \mathcal{R} is a $(G \times \mathcal{S})$ -invariant associative noncommutative product \circ satisfying (4.4), (4.5) and (4.6) such that the bilinear pairing $\mathcal{Q}(\phi, \psi) = \mathcal{T}(\phi \circ \psi)$ on \mathcal{R} is graded supersymmetric and non-degenerate.

A perfect Dixmier product makes \mathcal{R} into a filtered superalgebra which then, together with the Hamiltonian functions ϕ^x , $x \in \mathfrak{g}$, and α , is a perfect Dixmier algebra for \mathcal{M} in the sense of Definition 2.1. Conversely, we have shown that a perfect Dixmier algebra for \mathcal{M} yields a perfect Dixmier product on \mathcal{R} .

Remark 4.4. We can enrich our axiom set to get the notions for \mathcal{M} of a *positive Dixmier algebra* and a *positive Dixmier product*. The extra axioms require lifting the complex conjugation automorphism σ of \mathcal{O} (induced by the Cartan involution of \mathfrak{g}) to an antiholomorphic automorphism $\tilde{\sigma}$ of \mathcal{M} (of order 2 or 4) such that (i) $\tilde{\sigma}$ induces a \mathbb{C} -antilinear \circ -algebra automorphism of \mathcal{R} and (ii) the Hermitian pairing $(\phi|\psi) = \mathcal{T}(\phi \circ \psi \tilde{\sigma})$ is positive-definite. (\mathcal{T} being a supertrace is equivalent to this pairing being Hermitian.) We develop this in [B2] in the context of star products.

5. GRADED STAR PRODUCTS

Let $\mathcal{A} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} \mathcal{A}^j$ be a graded Poisson algebra as in [B1, Definition 2.2.1]. Assume $\mathcal{A}^0 = \mathbb{C}$. Then a *graded star product* (with parity) on \mathcal{A} is a product \star on $\mathcal{A}[t]$ which makes $\mathcal{A}[t]$ into an associative algebra over $\mathbb{C}[t]$ such that, for $\phi, \psi \in \mathcal{A}$, the series $\phi \star \psi = \sum_{p=0}^{\infty} C_p(\phi, \psi) t^p$ satisfies:

- (i) $C_0(\phi, \psi) = \phi\psi$
- (ii) $C_1(\phi, \psi) = \frac{1}{2}\{\phi, \psi\}t$
- (iii) $C_p(\phi, \psi) = (-1)^p C_p(\psi, \phi)$
- (iv) $C_p(\phi, \psi) \in \mathcal{A}^{j+k-p}$ when $\phi \in \mathcal{A}^j$ and $\psi \in \mathcal{A}^k$

We do *not* require that \star is bidifferential, i.e. that the operators $C_p(\cdot, \cdot)$ are bidifferential.

A graded star product \star on \mathcal{A} specializes at $t = 1$ to give a noncommutative product \circ on \mathcal{A} . Clearly \circ uniquely determines \star . Let $T : \mathcal{A} \rightarrow \mathcal{A}^0 = \mathbb{C}$ be the projection defined by the grading. Then we have a bilinear pairing $\mathcal{Q} : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}$ defined by

$$\mathcal{Q}(\phi, \psi) = T(\phi \circ \psi) \tag{5.1}$$

Notice that $\mathcal{Q}(\phi, \psi) = C_{2k}(\phi, \psi)$ if $\phi, \psi \in \mathcal{A}^k$ and so the parity axiom (iii) implies that \mathcal{Q} is symmetric if $k \in \mathbb{N}$ or anti-symmetric if $k \in \mathbb{N} + \frac{1}{2}$.

Definition 5.1. We say \star is *orthogonally graded* if \mathcal{A}^j and \mathcal{A}^k are \mathcal{Q} -orthogonal when $j \neq k$. We say \star is *perfectly graded* if also \mathcal{Q} is non-degenerate on \mathcal{A}^j for each j .

Comparing with the proof of Proposition 4.1, we find

Lemma 5.2. *If \star is orthogonally graded then T is a supertrace on \mathcal{A} with respect to \circ . If \star is perfectly graded then*

$$\mathcal{A}^j \star \mathcal{A}^k \subset \mathcal{A}^{j+k} \oplus t\mathcal{A}^{j+k-1} \oplus \dots \oplus t^{2 \min(j,k)} \mathcal{A}^{|j-k|} \quad (5.2)$$

Suppose we have Hamiltonian symmetry $\phi : \mathfrak{g} \rightarrow \mathcal{A}^1$, $x \mapsto \phi^x$, as in [B1, Definition 3.1.1] Put $[\phi, \psi]_\star = \phi \star \psi - \psi \star \phi$. We say \star is \mathfrak{g} -covariant if $[\phi^x, \phi^y]_\star = t\phi^{[x,y]}$ for all $x, y \in \mathfrak{g}$. We say \star is *exactly \mathfrak{g} -invariant* (or *strongly \mathfrak{g} -invariant*) if we have the stronger property:

$$[\phi^x, \psi]_\star = t\{\phi^x, \psi\} \quad (5.3)$$

for all $x \in \mathfrak{g}$ and $\psi \in \mathcal{A}$. Exact \mathfrak{g} -invariance implies ordinary G -invariance, i.e., $(g \cdot \psi_1) \star (g \cdot \psi_2) = g \cdot (\psi_1 \star \psi_2)$ where G acts on $\mathcal{A}[t]$ by $g \cdot (\psi t^p) = (g \cdot \psi) t^p$.

Lemma 5.3. *If \star is an orthogonally graded star product on \mathcal{A} then \star is exactly \mathcal{A}^1 -invariant.*

Proof. If $\phi \in \mathcal{A}^1$ and $\psi \in \mathcal{A}$ then $\phi \star \psi = \phi\psi + \frac{1}{2}\{\phi, \psi\}t + t^2 C_2(\phi, \psi)$. Then $[\phi^x, \psi]_\star = t\{\phi^x, \psi\}$ because of the parity axiom (iii). \square

Lemma 5.4. *Suppose \circ is a perfect Dixmier product on \mathcal{R} . Then \circ is the specialization at $t = 1$ of a unique graded star product \star on \mathcal{R} . Moreover \star is perfectly graded and \mathcal{S} -invariant.*

Conversely, suppose \star is a perfectly graded, \mathcal{S} -invariant star product on \mathcal{R} . Then the specialization at $t = 1$ of \star is a perfect Dixmier product on \mathcal{R} .

Proof. Given \circ , we define a product \star on $\mathcal{R}[t]$ as follows: \star is $\mathbb{C}[t]$ -bilinear and if $\phi \in \mathcal{R}^j$ and $\psi \in \mathcal{R}^k$ with $\phi \circ \psi = \sum_{i=|j-k|}^{j+k} \pi_i$ where $\pi_i \in \mathcal{R}^i$ then $\phi \star \psi = \sum_{i=|j-k|}^{j+k} \pi_i t^{j+k-i}$. This is the only possible way to extend \circ to a graded star product. The properties of \circ imply that \star is in fact a perfectly graded, \mathcal{S} -invariant star product. The converse is clear. \square

5.1. The operators Λ^x . Here is an important consequence of perfectness.

Proposition 5.5. *Suppose we have a perfect Dixmier product \circ on \mathcal{R} . For $x \in \mathfrak{g}$ and any $\psi \in \mathcal{R}$ we have*

$$\phi^x \circ \psi = \phi^x \psi + \frac{1}{2}\{\phi^x, \psi\} + \Lambda^x(\psi) \quad (5.4)$$

where $\Lambda^x : \mathcal{R} \rightarrow \mathcal{R}$ are linear operators. These satisfy

- (i) Λ^x is the \mathcal{Q} -adjoint of ordinary multiplication by ϕ^x .
- (ii) If $x \neq 0$ and j is positive, then Λ^x is non-zero somewhere on \mathcal{R}^j .
- (iii) Λ^x is graded of degree -1 , i.e., $\Lambda^x(\mathcal{R}^j) \subseteq \mathcal{R}^{j-1}$.
- (iv) The operators Λ^x commute, i.e., $[\Lambda^x, \Lambda^y] = 0$ for all $x, y \in \mathfrak{g}$.
- (v) The operators Λ^x transform in the adjoint representation of \mathfrak{g} , i.e., $[\Phi^x, \Lambda^y] = \Lambda^{[x,y]}$ where $\Phi^x = \{\phi^x, \cdot\}$.
- (vi) The operators Λ^x commute with the \mathcal{S} -action on \mathcal{R} .

Proof. Same as the proof of [B1, Corollary 8.4.1]. \square

If we identify \mathcal{R} with \mathcal{D} via \mathbf{q} , then we get the representation

$$\Pi : \mathfrak{g} \oplus \mathfrak{g} \rightarrow \text{End}_{\mathcal{S}} \mathcal{R}, \quad \Pi^{(x,y)}(\psi) = \phi^x \circ \psi - \psi \circ \phi^y \quad (5.5)$$

Then Proposition 5.5 gives

Corollary 5.6. *For $x \in \mathfrak{g}$ we have $\Pi^{(x,x)} = \eta^x$ and $\Pi^{(x,-x)} = 2\phi^x + 2\Lambda^x$.*

REFERENCES

- [B1] R. Brylinski, *Quantization of double covers of nilpotent coadjoint orbits I: noncommutative models*, preprint, posted on Math Front.
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